Sequences

A **sequence** is a function whose domain is the set of positive integers.

If the domain of $f(x)=3x-2$ is limited to positive integers, the sequence 1, 4, 7, 10, 13… is generated.

We write the above sequence using **sequence notation** as $\{a_n\}=\{3n-2\}$. Here, a is a term in the sequence, and n is the term number. For example, the tenth term in this sequence would be given by $a_{10}=3(10)-2=28$.

Examples: Write the first five terms of each sequence:

a) $\{s_n\}=\{n^2+1\}$

$s_1=1^2+1=2$
$s_2=2^2+1=5$
$s_3=3^2+1=10$
$s_4=4^2+1=17$
$s_5=5^2+1=26$

b) $\{c_n\}=\{(-1)^{n+1}\cdot n^2\}$

$c_1=(-1)^{1+1}\cdot 1^2=1$
$c_2=(-1)^{2+1}\cdot 2^2=-4$
$c_3=(-1)^{3+1}\cdot 3^2=9$
$c_4=(-1)^{4+1}\cdot 4^2=-16$
$c_5=(-1)^{5+1}\cdot 5^2=25$

c) $\{t_n\}=\left\{\dfrac{(-1)^n}{(n+1)(n+2)}\right\}$

$t_1=\dfrac{(-1)^1}{(1+1)(1+2)}=-\dfrac{1}{6}$
$t_2=\dfrac{(-1)^2}{(2+1)(2+2)}=\dfrac{1}{12}$
$t_3=\dfrac{(-1)^3}{(3+1)(3+2)}=-\dfrac{1}{20}$
$t_4=\dfrac{(-1)^4}{(4+1)(4+2)}=\dfrac{1}{30}$
$t_5=\dfrac{(-1)^5}{(5+1)(5+2)}=-\dfrac{1}{42}$

Note: The term $(-1)^n$ makes the odd terms of a sequence negative and the even terms positive. The terms $(-1)^{n+1}$ or $(-1)^{n-1}$ make the even terms negative and the odd terms positive.

Examples: Write down the nth term of the sequence suggested by each pattern.

a) $-\dfrac{2}{3},\dfrac{4}{9},-\dfrac{8}{27},\dfrac{16}{81},\ldots$ numerators are powers of 2, denominators are powers of 3. Alternating signs.

n	a_n
1	$-2/3$
2	$4/9$
3	$-8/27$
4	$16/81$
⋮	
n	$(-1)^n\cdot\dfrac{2^n}{3^n}$

$a_n=(-1)^n\cdot\dfrac{2^n}{3^n}$

$\boxed{a_n=\left(-\dfrac{2}{3}\right)^n}$

b) $1,\dfrac{1}{2},3,\dfrac{1}{4},5,\dfrac{1}{6},7,\dfrac{1}{8},\ldots$

n	a_n
1	1
2	1/2
3	3
4	1/4
5	5
6	1/6
7	7
8	1/8

$\boxed{a_n=\begin{cases} n & \text{if } n \text{ is odd} \\ 1/n & \text{if } n \text{ is even}\end{cases}}$

The Factorial Symbol

If $n\ge 0$ is an integer, then the symbol $n!$ is defined as follows:

$0!=1$ $\quad 1!=1$ $\quad 2!=2\cdot 1=2$ $\quad 3!=3\cdot 2\cdot 1=6$ $\quad 4!=4\cdot 3\cdot 2\cdot 1=24$

$n!=n\cdot(n-1)\cdot\ldots\cdot 3\cdot 2\cdot 1$ $\qquad n!=n\cdot(n-1)!$

Examples: Evaluate the following:

a) $6!=6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1=\boxed{720}$

b) $\dfrac{7!}{4!}=\dfrac{7\cdot 6\cdot 5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$
$=7\cdot 6\cdot 5=\boxed{210}$

c) $\dfrac{8!}{6!2!}=\dfrac{8\cdot 7\cdot\cancel{6!}}{\cancel{6!}\,2!}=\dfrac{8\cdot 7}{2}=\boxed{28}$

Recursive Formulas

A recursively-defined sequence assigns a value to the first term or first few terms, then specifies the nth term by a formula or equation that involves one or more of the terms preceding it.

Examples: Write the first five terms of each recursively-defined sequence.

a) $a_1 = 3,\ a_n = 4 - a_{n-1}$

$a_1 = 3$
$a_2 = 4 - a_1 = 4 - 3 = 1$
$a_3 = 4 - a_2 = 4 - 1 = 3$
$a_4 = 4 - a_3 = 4 - 3 = 1$
$a_5 = 4 - a_4 = 4 - 1 = 3$

b) $a_1 = -2,\ a_n = n + 3a_{n-1}$

$a_1 = -2$
$a_2 = 2 + 3a_1 = 2 + 3(-2) = -4$
$a_3 = 3 + 3a_2 = 3 + 3(-4) = -9$
$a_4 = 4 + 3a_3 = 4 + 3(-9) = -23$
$a_5 = 5 + 3a_4 = 5 + 3(-23) = -64$

c) $a_1 = -1,\ a_2 = 1,\ a_n = a_{n-2} + na_{n-1}$

$a_1 = -1$
$a_2 = 1$
$a_3 = a_1 + 3a_2 = -1 + 3(1) = 2$
$a_4 = a_2 + 4a_3 = 1 + 4(2) = 9$
$a_5 = a_3 + 5a_4 = 2 + 5(9) = 47$

d) $a_1 = A,\ a_n = ra_{n-1}$

$a_1 = A$
$a_2 = ra_1 = rA$
$a_3 = ra_2 = r(rA) = r^2A$
$a_4 = ra_3 = r(r^2A) = r^3A$
$a_5 = ra_4 = r(r^3A) = r^4A$

Summation Notation

The symbol Σ (the Greek letter sigma) is an instruction to add up the terms of a sequence. The integer k is called the **index** of the sum; it tells you where to start the sum and where to end it. The expression $\sum_{k=1}^{n} a_k$ is an instruction to add the terms a_k of the sequence $\{a_n\}$ starting with $k = 1$ and ending with $k = n$. It is read, "the sum of a_k from $k = 1$ to $k = n$."

Examples: Express each sum using summation notation.

a) $1 + 3 + 5 + 7 + \ldots + [2(12) - 1]$

$[2(1)-1] + [2(2)-1] + [2(3)-1] + [2(4)-1] + \ldots + [2(12)-1]$

$\boxed{\sum_{k=1}^{12} 2k-1}$

b) $\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \cdots + \frac{n}{e^n}$

$\boxed{\sum_{k=1}^{n} \frac{k}{e^k}}$

Examples: Write out each sum.

a) $\sum_{k=1}^{n} (k+1)^2 =$

$(1+1)^2 + (2+1)^2 + (3+1)^2 + \ldots + (n+1)^2 =$

$2^2 + 3^2 + 4^2 + \ldots + (n+1)^2 =$

$\boxed{4 + 9 + 16 + \ldots + (n+1)^2}$

b) $\sum_{k=0}^{n-1} \left(\frac{1}{3^{k+1}}\right) = \frac{1}{3^{0+1}} + \frac{1}{3^{1+1}} + \frac{1}{3^{2+1}} + \ldots + \frac{1}{3^{n-1+1}}$

$= \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^n}$

$= \boxed{\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^n}}$

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences, and c is a real number, then:

- $\sum_{k=1}^{n} (ca_k) = ca_1 + ca_2 + \ldots + ca_n = c(a_1 + a_2 + \ldots + a_n) = c\sum_{k=1}^{n} a_k$
- $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
- $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$
- $\sum_{k=j}^{n} a_k = \sum_{k=1}^{n} a_k - \sum_{k=1}^{j-1} a_k$, where $1 < j < n$.

Formulas for Sums of Sequences

- $\sum_{k=1}^{n} c = \underbrace{c+c+\ldots+c}_{n \text{ times}} = cn$
- $\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Examples: Find the sum of each sequence.

a) $\sum_{k=1}^{5}(5k) = 5\sum_{k=1}^{5} k$

$= 5\left[\frac{5(5+1)}{2}\right]$

$= 5(15) = \boxed{75}$

b) $\sum_{k=1}^{26}(3k-7) = 3\sum_{k=1}^{26} k - \sum_{k=1}^{26} 7$

$= 3\left[\frac{26(26+1)}{2}\right] - 7(26)$

$= 3(351) - 182$

$= \boxed{871}$

c) $\sum_{k=1}^{14}(k^2-4) = \sum_{k=1}^{14} k^2 - \sum_{k=1}^{14} 4$

$= \frac{14(14+1)(2\cdot 14+1)}{6} - 4(14)$

$= 1015 - 56 = \boxed{959}$

d) $\sum_{k=8}^{40}(-3k) = -3\sum_{k=8}^{40} k = -3\left(\sum_{k=1}^{40} k - \sum_{k=1}^{7} k\right)$

$= -3\left[\frac{40(40+1)}{2} - \frac{7(7+1)}{2}\right] = -3(820-28)$

$= \boxed{-2376}$

e) $\sum_{k=4}^{24} k^3 = \sum_{k=1}^{24} k^3 - \sum_{k=1}^{3} k^3$

$= \left(\frac{24(24+1)}{2}\right)^2 - \left(\frac{3(3+1)}{2}\right)^2$

$= 300^2 - 6^2 = \boxed{89{,}964}$

Math 1050 – 9.2 Notes

Arithmetic Sequences

An **arithmetic sequence** is one in which the difference between successive terms of a sequence is always the same number.

An arithmetic sequence may be defined recursively as $a_1 = a,\ a_n = a_{n-1} + d$, where a is the first term and d is the **common difference.**

The terms of an arithmetic sequence with first term a_1 and common difference d follow the pattern $a_1,\ a_1 + d,\ a_1 + 2d,\ a_1 + 3d, \ldots,\ a_1 + (n-1)d$, where $d = a_n - a_{n-1}$.

Examples: Determine whether the following sequences are arithmetic:

a) 3, 7, 11, 15, 19,...

$7-3=4$
$11-7=4$
$15-11=4$
$19-15=4$

$\boxed{\text{yes}, d=4}$

b) $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, \ldots$

$1-\frac{1}{3}=\frac{2}{3}$
$\frac{5}{3}-1=\frac{2}{3}$
$\frac{7}{3}-\frac{5}{3}=\frac{2}{3}$

$\boxed{\text{yes}, d=\frac{2}{3}}$

Examples: Show that the following sequences are arithmetic and find the common difference.

a) $\{s_n\} = \{2n-4\}$

$s_n = 2n-4$
$s_{n-1} = 2(n-1)-4 = 2n-2-4 = 2n-6$
$s_n - s_{n-1} = (2n-4)-(2n-6)$
$= 2n-4-2n+6 = 2$

$\boxed{\text{arithmetic}, d=2}$

b) $\{b_n\} = \{\ln 2^n\}$

$b_n = \ln 2^n = n\ln 2$
$b_{n-1} = \ln 2^{n-1} = (n-1)\ln 2 = n\ln 2 - \ln 2$
$b_n - b_{n-1} = n\ln 2 - (n\ln 2 - \ln 2)$
$= n\ln 2 - n\ln 2 + \ln 2 = \ln 2$

$\boxed{\text{arithmetic}, d=\ln 2}$

***n*th Term of an Arithmetic Sequence:** For an arithmetic sequence $\{a_n\}$ whose first term is a_1 and whose common difference is d, the nth term is determined by the formula $a_n = a_1 + (n-1)d$.

Examples: Find the *n*th term and the fifty-first term of the following sequences.

a) $a_1 = 6,\ d = -2$

$a_n = 6 + (n-1)(-2)$
$a_n = 6 - 2n + 2$
$\boxed{a_n = -2n+8}$
$a_{51} = -2(51)+8$
$\boxed{a_{51} = -94}$

b) $a_1 = 1,\ d = -1/3$

$a_n = 1 + (n-1)(-\frac{1}{3})$
$a_n = 1 + (-\frac{1}{3}n + \frac{1}{3})$
$\boxed{a_n = -\frac{1}{3}n + \frac{4}{3}}$
$a_{51} = -\frac{1}{3}(51) + \frac{4}{3} = -\frac{51}{3} + \frac{4}{3}$
$\boxed{a_{51} = -\frac{47}{3}}$

Examples: Find the indicated term in each arithmetic sequence.

a) 80th term of -1, 1, 3, …

$a_1 = -1 \quad d = 2 \quad n = 80$
$a_n = a_1 + (n-1)d$
$a_{80} = -1 + (80-1)(2)$
$= \boxed{157}$

b) 86th term of $2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

$a_1 = 2 \quad d = \frac{1}{2} \quad n = 86$
$a_n = a_1 + (n-1)d$
$a_{86} = 2 + (86-1)(\frac{1}{2})$
$= 2 + \frac{85}{2} = \boxed{\frac{89}{2}}$

Examples: Find the first term and common difference of the arithmetic sequence described. Give a recursive formula for the sequence, and write a formula for the *n*th term.

a) 4th term is 3, 20th term is 35

$a_4 = a_1 + (4-1)d$

$3 = a_1 + 3d$

$a_{20} = a_1 + (20-1)d$

$35 = a_1 + 19d$

$(-1)(a_1 + 3d = 3)$

$a_1 + 19d = 35$

$-a_1 - 3d = -3$

$16d = 32$

$\boxed{d = 2}$

$a_1 + 3(2) = 3$

$\boxed{a_1 = -3}$

$\boxed{a_1 = -3 \quad a_n = a_{n-1} + 2}$

$a_n = -3 + (n-1)(2)$

$a_n = -3 + 2n - 2$

$\boxed{a_n = 2n - 5}$

b) 5th term is -2, 13th term is 30

$a_5 = a_1 + (5-1)d$

$-2 = a_1 + 4d$

$a_{13} = a_1 + (13-1)d$

$30 = a_1 + 12d$

$(-1)(a_1 + 4d = -2)$

$a_1 + 12d = 30$

$-a_1 - 4d = 2$

$8d = 32$

$\boxed{d = 4}$

$a_1 + 4(4) = -2$

$\boxed{a_1 = -18}$

$\boxed{a_1 = -18 \quad a_n = a_{n-1} + 4}$

$a_n = -18 + (n-1)(4)$

$a_n = -18 + 4n - 4$

$\boxed{a_n = 4n - 22}$

Sum of an Arithmetic Sequence

The sum S_n of the first n terms of an arithmetic sequence $\{a_n\}$ with first term a_1 and common difference d is given by $S_n = a_1 + a_2 + a_3 + \ldots + a_n = \frac{n}{2}(a_1 + a_n)$.

Examples: Find each sum.

a) $-1 + 3 + 7 + \ldots + (4n - 5)$

$a_1 = -1 \quad a_n = 4n - 5$

$S_n = \frac{n}{2}(-1 + 4n - 5) = \frac{n}{2}(4n - 6)$

$= \frac{4n^2 - 6n}{2} = \boxed{2n^2 - 3n}$

b) $1 + 3 + 5 + \ldots + 59$

$a_1 = 1 \quad d = 2 \quad a_n = 59 \quad n = ?$

$a_n = a_1 + (n-1)d$

$59 = 1 + (n-1)(2)$

$59 = 1 + 2n - 2$

$59 = 2n - 1$

$60 = 2n$

$n = 30$

$S_{30} = \frac{30}{2}(1 + 59)$

$= \boxed{900}$

c) $7 + 1 - 5 - 11 - \ldots - 299$

$a_1 = 7 \quad d = -6 \quad a_n = -299 \quad n = ?$

$-299 = 7 + (n-1)(-6)$

$-299 = 7 - 6n + 6$

$-299 = -6n + 13$

$-312 = -6n$

$n = 52$

$S_{52} = \frac{52}{2}(7 + (-299))$

$= \boxed{-7592}$

d) $\sum_{n=1}^{90}(3 - 2n)$

$a_1 = 3 - 2(1) = 1$

$a_{90} = 3 - 2(90) = -177$

$S_{90} = \frac{90}{2}(1 + (-177)) = \boxed{-7920}$

e) $\sum_{n=1}^{80}\left(\frac{n}{3} + \frac{1}{2}\right)$

$a_1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

$a_{80} = \frac{80}{3} + \frac{1}{2} = \frac{163}{6}$

$S_{80} = \frac{80}{2}\left(\frac{5}{6} + \frac{163}{6}\right) = \boxed{1120}$

Example: The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?

$a_1 = 15 \quad n = 40 \quad d = 2$

$a_{40} = 15 + (40-1)(2) = 93$

$S_{40} = \frac{40}{2}(15 + 93) = \boxed{2160}$

Geometric Sequences and Geometric Series

A **geometric sequence** is one in which the ratio of successive terms is always the same nonzero number.

A **geometric sequence** may be defined recursively as $a_1 = a,\ a_n = ra_{n-1}$, where a is the first term and $r \neq 0$ is the **common ratio.**

The terms of a geometric sequence with first term a_1 and common ratio $r = \frac{a_n}{a_{n-1}}$ follow the pattern

$a_1,\ a_1r,\ a_1r^2,\ a_1r^3, \dots, a_1r^{n-1}$.

Examples: Find the common ratio of each geometric sequence and write out the first four terms.

a) $\{b_n\} = \{(-5)^n\}$ $\quad b_n = (-5)^n$ $\quad b_{n-1} = (-5)^{n-1}$

$r = \frac{b_n}{b_{n-1}} = \frac{(-5)^n}{(-5)^{n-1}} = (-5)^{n-(n-1)} = \boxed{-5}$

$b_1 = (-5)^1 = -5$

$b_2 = (-5)^2 = 25$

$b_3 = (-5)^3 = -125$

$b_4 = (-5)^4 = 625$

b) $\{u_n\} = \left\{\frac{2^n}{3^{n-1}}\right\}$ $\quad U_n = \frac{2^n}{3^{n-1}}$ $\quad U_{n-1} = \frac{2^{n-1}}{3^{n-2}}$

$r = \frac{U_n}{U_{n-1}} = \frac{2^n/3^{n-1}}{2^{n-1}/3^{n-2}} = \frac{2^n}{3^{n-1}} \cdot \frac{3^{n-2}}{2^{n-1}} = \boxed{\frac{2}{3}}$

$U_1 = 2^1/3^{1-1} = 2/1 = 2$

$U_2 = 2^2/3^{2-1} = 4/3$

$U_3 = 2^3/3^{3-1} = 8/9$

$U_4 = 2^4/3^{4-1} = 16/27$

Examples: Determine whether the given sequence is arithmetic, geometric, or neither. If it is arithmetic, give the common difference. If it is geometric, give the common ratio. Arithmetic if $a_n - a_{n-1}$ is constant

Geometric if a_n/a_{n-1} is constant

a) $\{5n^2+1\}$ $\quad a_n = 5n^2+1$

$a_{n-1} = 5(n-1)^2+1$

$= 5(n^2-2n+1)+1$

$= 5n^2-10n+5+1$

$= 5n^2-10n+6$

$a_n - a_{n-1} = (5n^2+1)-(5n^2-10n+6)$

$= 5n^2+1-5n^2+10n-6$

$= 10n-5$ $\boxed{\text{not arithmetic}}$

$\frac{a_n}{a_{n-1}} = \frac{5n^2+1}{5n^2-10n+6}$ $\boxed{\text{not geometric}}$

b) $\left\{\left(\frac{5}{4}\right)^n\right\}$ $\quad a_n = \left(\frac{5}{4}\right)^n$ $\quad a_{n-1} = \left(\frac{5}{4}\right)^{n-1}$

$a_n - a_{n-1} = \left(\frac{5}{4}\right)^n - \left(\frac{5}{4}\right)^{n-1}$ $\boxed{\text{not arithmetic}}$

$\frac{a_n}{a_{n-1}} = \frac{(5/4)^n}{(5/4)^{n-1}} = \left(\frac{5}{4}\right)^{n-(n-1)} = \frac{5}{4}$

$\boxed{\text{geometric, } r = 5/4}$

***n*th Term of a Geometric Sequence:** For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r, the nth term is determined by the formula $a_n = a_1r^{n-1};\ r \neq 0$.

Examples: Find the *n*th term and the 5th term of the geometric sequence.

a) $a_1 = -2,\ r = 4$

$\boxed{a_n = -2(4)^{n-1}}$

$a_5 = -2(4)^{5-1}$

$= -2(4)^4$

$= -2(256) = \boxed{-512}$

b) $a_1 = 1,\ r = -\frac{1}{3}$

$a_n = 1\left(-1/3\right)^{n-1}$

$\boxed{a_n = \left(-1/3\right)^{n-1}}$

$a_5 = \left(-1/3\right)^{5-1} = \left(-1/3\right)^4 = \boxed{1/81}$

Examples: Find the indicated term of each geometric sequence. $a_n = a_1 \cdot r^{n-1}$

a) 8th term of 1, 3, 9, …

$a_1 = 1 \quad r = 3/1 = 3$

$a_8 = 1 \cdot 3^{8-1} = 3^7 = \boxed{2187}$

b) 7th term of 9, –6, 4, …

$a_1 = 9 \quad r = -6/9 = -2/3$

$a_7 = 9 \cdot (-2/3)^{7-1} = 9 \cdot (-2/3)^6$

$= 9\,(64/729) = \boxed{64/81}$

Examples: Find the *n*th term of each geometric sequence.

a) 5, 10, 20, 40, …

$a_1 = 5 \quad r = 10/5 = 2$

$\boxed{a_n = 5 \cdot 2^{n-1}}$

b) $a_2 = 7,\ r = 1/3$

$a_1 = 7 \div 1/3 = 21$

$\boxed{a_n = 21 \cdot (1/3)^{n-1}}$

c) $a_3 = 1/3,\ a_6 = 1/81$

$1/3 = a_1 \cdot r^{3-1}$

$1/3 = a_1 \cdot r^2$

$1/81 = a_1 \cdot r^{6-1}$

$1/81 = a_1 \cdot r^5$

$\frac{1/81}{1/3} = \frac{a_1 r^5}{a_1 r^2}$

$1/27 = r^3$

$r = \sqrt[3]{1/27} = 1/3$

$1/3 = a_1 \cdot (1/3)^2$

$1/3 = 1/9\, a_1$

$a_1 = 3$

$a_n = 3 \cdot \left(\frac{1}{3}\right)^{n-1}$

$a_n = \left(\frac{1}{3}\right)^{-1} \cdot \left(\frac{1}{3}\right)^{n-1}$

$\boxed{a_n = \left(\frac{1}{3}\right)^{n-2}}$

Sum of the First *n* Terms of a Geometric Sequence

The sum S_n of the first n terms of a geometric sequence $\{a_n\}$ with first term a_1 and common ratio r is given by $S_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 \cdot \frac{1-r^n}{1-r}$, where $r \neq 0, 1$.

Examples: Find each sum.

a) $\frac{3}{9} + \frac{3^2}{9} + \frac{3^3}{9} + \ldots + \frac{3^n}{9}$

$a_1 = 3/9 = 1/3 \quad r = \frac{3^2/9}{3/9} = 3$

$S_n = 1/3 \cdot \frac{1-3^n}{1-3} = \frac{1}{3} \cdot \frac{1-3^n}{-2}$

$\boxed{S_n = \frac{1-3^n}{-6} = \frac{3^n - 1}{6}}$

b) $\sum_{k=1}^{15} 4 \cdot 3^{k-1}$

$a_1 = 4 \cdot 3^{1-1} = 4 \quad r = 3$

$S_{15} = 4 \cdot \frac{1-3^{15}}{1-3} = \boxed{28{,}697{,}812}$

c) $2 + \frac{6}{5} + \frac{18}{25} + \ldots + 2\left(\frac{3}{5}\right)^{15}$

$2(3/5)^0$

16^{th} term

$a_1 = 2 \quad r = 3/5 \quad n = 16$

$S_{16} = 2 \cdot \frac{1-(3/5)^{16}}{1-(3/5)} = \boxed{4.9986}$

d) $\sum_{n=1}^{5} \left(\frac{2}{3}\right)^n = \sum_{n=1}^{5} \frac{2}{3} \cdot \left(\frac{2}{3}\right)^{n-1}$

$a_1 = 2/3 \quad r = 2/3$

$S_5 = \frac{2}{3} \cdot \frac{1-(2/3)^5}{1-(2/3)} = \boxed{\frac{422}{243}}$

Infinite Series: An infinite geometric series is the sum of the terms of an infinite geometric sequence. It is denoted by $\sum_{k=1}^{\infty} a_1 r^{k-1}$. If the sum of the first n terms of the geometric sequence, S_n, approaches a number L as $n \to \infty$, we say that the infinite geometric series **converges** and that its sum is L. If a series does not converge, it is a **divergent series.**

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is $S = \frac{a_1}{1-r}$.

Examples: Determine whether each geometric series converges or diverges. If it converges, find its sum.

a) $2+\frac{4}{3}+\frac{8}{9}+\ldots$ $r=4/3 \div 2 = 2/3$

$|r|<1$ $\boxed{\text{converges}}$

$S=\frac{2}{1-2/3}=\boxed{6}$

b) $9+12+16+\frac{64}{3}+\ldots$ $r= 12/9 = 4/3$

$|r|>1$ $\boxed{\text{diverges}}$

c) $1-\frac{3}{4}+\frac{9}{16}-\frac{27}{64}+\ldots$ $r=-\frac{3}{4}\div 1 = -\frac{3}{4}$

$|r|<1$ $\boxed{\text{converges}}$

$S=\frac{1}{1-(-3/4)}=\boxed{4/7}$

d) $\sum_{k=1}^{\infty} 3\left(\frac{3}{2}\right)^{k-1}$

$r=3/2$

$|r|>1$ $\boxed{\text{diverges}}$

e) $\sum_{k=1}^{\infty} 2\left(\frac{3}{4}\right)^{k} = \sum_{k=1}^{\infty} 2\cdot\frac{3}{4}\cdot\left(\frac{3}{4}\right)^{k-1}$

$= \sum_{k=1}^{\infty} \frac{3}{2}\cdot\left(\frac{3}{4}\right)^{k-1}$

$a_1 = 3/2$ $r=3/4$

$|r|<1$ $\boxed{\text{converges}}$

$S=\frac{3/2}{1-3/4}=\boxed{6}$

The Binomial Theorem

Binomial Coefficients (also known as ***combinations***): $\binom{n}{j} = \dfrac{n!}{j!(n-j)!}$. The symbol $\binom{n}{j}$ is read "n choose j."

It can also be written as $C(n,j)$, C_j^n, or ${}_nC_j$. This expression is most commonly used in probability. It is the number of ways of selecting distinct groups of j objects from a group of n objects. For example, "How many different groups of 5 students can be selected from a class of 30 students?"

Examples: Evaluate the following expressions.

a. $\binom{5}{2} = \dfrac{\cancel{5!}\ 5\cdot 4}{2!\,\cancel{3!}} = \dfrac{5\cdot4}{2\cdot1} = \boxed{10}$

b. $\binom{9}{3} = \dfrac{\cancel{9!}\ 9\cdot8\cdot7}{3!\,\cancel{6!}} = \dfrac{9\cdot8\cdot7}{3\cdot2\cdot1} = \boxed{84}$

c. $\binom{27}{1} = \dfrac{\cancel{27!}\ 27}{1!\,\cancel{26!}} = \dfrac{27}{1} = \boxed{27}$

d. $\binom{50}{17} = \dfrac{50!}{17!\,33!}$

way too large to do by hand

Calculator gives $\boxed{9{,}847{,}379{,}391{,}150}$

Pascal's Triangle:

Row 0: 1
Row 1: 1 1
Row 2: 1 2 1
Row 3: 1 3 3 1
Row 4: 1 4 6 4 1
Row 5: 1 5 10 10 5 1
Row 6: 1 6 15 20 15 6 1
Row 7: 1 7 21 35 35 21 7 1
Row 8: 1 8 28 56 70 56 28 8 1
Row 9: 1 9 36 84 126 126 84 36 9 1
etc.

Pascal's triangle is a handy way of organizing binomial coefficients. The sides consist of all 1's, and any other entry is the sum of the two nearest entries in the row above. The top row (Row 0) consists of $\binom{0}{0}$, the next row $\binom{1}{0}$ and $\binom{1}{1}$, the next $\binom{2}{0}$, $\binom{2}{1}$, and $\binom{2}{2}$, etc. Row n consists of $\binom{n}{j}$ from $j=0$ to $j=n$.

Binomial Theorem: Let a and b be real numbers. For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{j}a^{n-j}b^j + \ldots + \binom{n}{n}b^n = \sum_{j=0}^{n}\binom{n}{j}a^{n-j}b^j$$

The coefficients are the values on row n of Pascal's triangle.

Examples: Use the binomial theorem to expand the following expressions:

a. $(x+4)^4$ 1 4 6 4 1

$= 1x^4 + 4x^3(4) + 6x^2(4)^2 + 4x(4)^3 + 1(4)^4$

$= \boxed{x^4 + 16x^3 + 96x^2 + 256x + 256}$

b. $(3x-2)^5$ 1 5 10 10 5 1

$= 1(3x)^5 + 5(3x)^4(-2) + 10(3x)^3(-2)^2 + 10(3x)^2(-2)^3 + 5(3x)(-2)^4 + 1(-2)^5$

$= \boxed{243x^5 - 810x^4 + 1080x^3 - 720x^2 + 240x - 32}$

c. $(x^2+3y)^3$ 1 3 3 1

$= 1(x^2)^3 + 3(x^2)^2(3y) + 3(x^2)(3y)^2 + 1(3y)^3$

$= \boxed{x^6 + 9x^4y + 9x^2y^2 + 27y^3}$

d. $(\sqrt{x}-\sqrt{3})^6$ 1 6 15 20 15 6 1

$= 1(\sqrt{x})^6 + 6(\sqrt{x})^5(-\sqrt{3}) + 15(\sqrt{x})^4(-\sqrt{3})^2 + 20(\sqrt{x})^3(-\sqrt{3})^3 + 15(\sqrt{x})^2(-\sqrt{3})^4 + 6(\sqrt{x})(-\sqrt{3})^5 + 1(-\sqrt{3})^6$

$= \boxed{x^3 - 6x^2\sqrt{3x} + 45x^2 - 20x\sqrt{3x} + 135x - 54\sqrt{3x} + 27}$

$(\sqrt{x})^5 = (\sqrt{x})^2 \cdot (\sqrt{x})^2 \cdot \sqrt{x}$ $\quad$ $(\sqrt{x})^3 = (\sqrt{x})^2 \cdot \sqrt{x}$

$= x \cdot x \cdot \sqrt{x} = x^2\sqrt{x}$ $\quad$ $= x\sqrt{x}$

Finding a Particular Coefficient or Term in a Binomial Expansion

Based on the expansion of $(a+b)^n$, the term containing a^j is $\binom{n}{n-j}a^j b^{n-j}$.

Examples:

a. Find the coefficient of x^5 in the expansion of $(x-1)^8$. $n=8$ $j=5$

$\binom{8}{3}x^5(-1)^3$ $\quad$ $\binom{8}{3} = \frac{8!}{3!5!} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1} = 56$

$= -56x^5$ $\quad$ $\boxed{-56}$

b. Find the coefficient of y^3 in the expansion of $(4y+2)^6$. $n=6$ $j=3$

$\binom{6}{3}(4y)^3(2)^3$ $\quad$ $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6\cdot5\cdot4}{3\cdot2\cdot1} = 20$

$= 20(64y^3)(8) = 10{,}240y^3$ $\quad$ $\boxed{10{,}240}$

c. Find the fifth term in the expansion of $(2x-y)^9$. x^9 x^8 x^7 x^6 $\underline{x^5}$

$n=9$ $j=5$

$\binom{9}{4}(2x)^5(-y)^4$ $\quad$ $\binom{9}{4} = \frac{9!}{4!5!} = \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1} = 126$

$= 126(32x^5)y^4 = \boxed{4032x^5y^4}$

d. Find the third term in the expansion of $(\sqrt{x}+1)^7$. $(\sqrt{x})^7$ $(\sqrt{x})^6$ $\underline{(\sqrt{x})^5}$

$n=7$ $j=5$

$\binom{7}{2}(\sqrt{x})^5(1)^2$ $\quad$ $\binom{7}{2} = \frac{7!}{2!5!} = \frac{7\cdot6}{2\cdot1} = 21$

$= \boxed{21x^2\sqrt{x}}$